Vortex dynamics in rotating counterflow and plane Couette and Poiseuille turbulence in superfluid helium

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(Received 14 March 2008; revised manuscript received 27 May 2008; published 31 July 2008)

An equation previously proposed to describe the evolution of vortex-line density in rotating counterflow turbulent tangles in superfluid helium [Phys. Rev B **69**, 094513 (2004)] is generalized to incorporate nonvanishing barycentric velocity and velocity gradients. Our generalization is compared with an analogous approach proposed by Lipniacki [Eur. J. Mech. B Fluids **25**, 435 (2006)], and with experimental results by Swanson *et al.* [Phys. Rev. Lett. **50**, 190 (1983)] in rotating counterflow, and it is used to evaluate the vortex density in plane Couette and Poiseuille flows of superfluid helium.

DOI: 10.1103/PhysRevB.78.024524 PACS number(s): 67.25.dk, 03.75.Lm

I. INTRODUCTION

Many researches of quantum vortices in superfluids have been carried out on rotating systems and counterflow situations, both of them with vanishing barycentric velocity gradient.[1–3] Evolution equations have been proposed to describe the influence of heat flux and of angular velocity on the vortex dynamics[4] generalizing the well-known Vinen's equation for nonrotating systems.[1–3,5] An interesting challenge is to generalize these vortex evolution equations to include the influence of barycentric flow, which has much practical interest, for instance, in cryogenic applications. Here, we carry out such a generalization and we examine a recent proposal by Lipniacki,[6] which opens an interesting perspective but which, on the other side, discloses some aspects which have not been yet settled out with enough clarity.

The aim of this paper is to generalize a previous equation proposed for rotating counterflow superfluid turbulence[4] by emphasizing more explicitly the dynamical role of the rotational of the superfluid velocity $\mathbf{v}_s$, related to quantized vortices. This allows us to write a proposal for the evolution equations of vortices in plane Couette and Poiseuille flows. In Sec. III we review some aspects of rotating counterflow and compare our generalized expression with Lipniacki's proposal,[6] which underlines the role of the polarization rather than of rot $\mathbf{v}_s$ itself, and we stress some open problems. In Sec. IV we use a thermodynamic formalism to relate the dynamical equation for vortices with a new term appearing in the mutual friction force, which we use for a comparison with that one by Lipniacki. In Sec. V we discuss several aspects of Couette and Poiseuille flows of superfluid helium including the presence of quantized vortices.

II. ROTATIONAL OF SUPERFLUID VELOCITY AND THE DYNAMICS OF VORTEX-LINE DENSITY

An evolution equation for the dynamics of quantum vortices in rotating helium under counterflow was proposed in Ref. 4 describing the influence of the heat flow and of angular velocity on the vortex-line density. In particular, the vortex-line density L was assumed to obey the following equation:

$$\frac{dL}{dt} = -\beta\kappa L^2 + [\alpha_1 V + \beta_2\sqrt{\kappa\Omega}]L^{3/2} - \left[\beta_1\Omega + \beta_4 V\sqrt{\frac{\Omega}{\kappa}}\right]L, \tag{2.1}$$

where β, α_1, β_2, β_1, and β_4 are dimensionless coefficients, $\kappa = h/m$ is the quantum of vorticity (m the mass of the ^{4}He atom and h Planck's constant), $V=|\mathbf{V}|$ (with $\mathbf{V}=\mathbf{v}_n-\mathbf{v}_s$) is the counterflow velocity, the relative velocity between averaged normal and superfluid velocities, which is proportional to the heat flux across the system, and $\Omega=|\mathbf{\Omega}|$ is the angular velocity of the container. Equation (2.1) mainly refers to the homogeneous situation which is reached after some transient time, otherwise a further term including vortex diffusion $\mathbf{J}^L$ has to be inserted. The right-hand side of Eq. (2.1), the so-called production term σ^L, reflects the variation of the vortex length L inside the system caused by reconnections. The original Vinen's equation corresponds to the first two terms. The other three terms incorporate the influence of the rotation in the simplest dimensionally consistent combinations. Terms in β_1 and β_2 reflect some competition between vortex formation and vortex reduction, as seen when $V=0$. The last coefficient expresses the interaction between rotation and counterflow, which makes that both effects are not merely additive. Its contribution could be interpreted by the fact that rotation tends to orientate and straighten out the vortex line along its direction, which makes unfavorable the existence of vortex line orthogonal to V. The sign of the several parameters is obtained by comparison with experimental data. Though apparently there are three new numerical coefficients, β_1, β_2, and β_4, only one of them is actually independent. Indeed, they are seen to satisfy the relations $\beta_4=\sqrt{2}\alpha_1$ and $\beta_1=\sqrt{2}\beta_2-2\beta$, which are required on relatively general arguments about the form of solutions. Their particular values were obtained in Ref. 4 by comparison with experimental data of Ref. 7. The values of the coefficients appearing in Eq. (2.1) were independently calculated in Ref. 8 and agree with those obtained in Ref. 4. When $\Omega=0$, Eq. (2.1) reduces to the well-known Vinen's equation,[5] with parameters α_1 and

β being, respectively, related to the production and destruction of vortices per unit volume and time.

In Refs. 4 and 9 it was shown that the value of coefficient α_1 depends on the angle between the counterflow velocity $\mathbf{V}$ and Schwarz's binormal vector $\mathbf{I}$ (Ref. 10) [see Eq. (3.6)]. As observed in Ref. 4 $\alpha_1 = \alpha_V \mathbf{I} \cdot \hat{\mathbf{V}}$, with α_V the coefficient appearing in Vinen's equation (pure counterflow).[1] Schwarz derived Vinen's equation using the vortex filament model obtaining $\alpha_V = \alpha c_1$, where α is the well-known coefficient appearing in the expression of the mutual friction force between vortex lines and the normal fluid and c_1 denote the average curvature of the tangle [see Eq. (3.7)]. In Ref. 4 in the regime of high rotation, the value $\mathbf{I} \cdot \hat{\mathbf{V}} = 1/2$ was found, so indicating that the vortex tangle is highly polarized. Coefficient β is linked to the average squared curvature of the vortices as $\beta\kappa = \alpha\tilde{\beta} c_2^2$, with c_2^2 defined in Eq. (3.7) and $\tilde{\beta}$ the vortex tension parameter, defined as $\epsilon_V = \kappa\rho_s\tilde{\beta}$, with ϵ_V the energy per unit length of vortex line.[1]

Equation (2.1) lacks an important source of vorticity, namely, a barycentric velocity gradient, which is known to produce turbulence in many actual flows. Thus, it would be useful to generalize Eq. (2.1) by incorporating in it barycentric velocity gradients. A possible way to do so would be simply adding new terms basing on dimensional analysis and on comparison with the observed phenomenology. Instead of proceeding in this way, we will interpret Eq. (2.1) in some deeper terms, which will be useful for a consistent incorporation of the velocity gradient.

To generalize Eq. (2.1) we note that, in the particular case of pure rotation, Ω is related to rot $\mathbf{v}_s$ as $2\Omega = |\text{rot } \mathbf{v}_s|$, where $\mathbf{v}_s$ is the macroscopic superfluid velocity. As Lipniacki[6] noted in a different proposal, writing an equation such as Eq. (2.1) in terms of rot $\mathbf{v}_s$ and V rather than in terms of Ω and V would be more general, because it would reduce to Eq. (2.1) for rotation, and it could be applied to other flows as plane Couette or Poiseuille flows (see Sec. V), where $|\text{rot } \mathbf{v}_s| = dv_{sx}(z)/dz$, x being the direction of the fluid motion, z the direction orthogonal to the parallel plates, and $v_{sx}(z)$ the macroscopical superfluid velocity, depending only on z.

The direct replacement of the quantity $2\Omega = |\text{rot } \mathbf{v}_s|$ in the Eq. (2.1) is not completely correct because whereas Ω is taken as an externally fixed parameter in Eq. (2.1), rot $\mathbf{v}_s$ is a dynamical quantity, which must be described by a suitable evolution equation. To overcome the problem we have to choose an evolution equation for the vortex-line density L which includes a vortex density flux $\mathbf{J}^L$ (Ref. 11),

$$\frac{\partial L}{\partial t} + \nabla \cdot \mathbf{J}^L = \sigma^L, \tag{2.2}$$

where σ_L stands for the production term generalizing the right-hand side of Eq. (2.1). Of course, the above equation takes into account also the inhomogeneities in the line density L inside the system. The form of $\mathbf{J}^L$ contains a convective contribution $L\mathbf{v}_\mathbf{L}$, with $\mathbf{v}_\mathbf{L}$ the velocity of vortex lines with respect to the laboratory frame, and a diffusive contribution. In some situations, when the rate of variation of the perturbations is higher than the reciprocal of the relaxation time of the diffusive flux,[11,12] one must take $\mathbf{J}^L$ as an independent variable.[13] Here, neglecting the relaxation time of $\mathbf{J}^L$ and considering isothermal situations, we take for $\mathbf{J}^L$ the following simple law, where the diffusive contribution is analogous to Fick's diffusion law:

$$\mathbf{J}^L = -\tilde{D}\nabla L + L\mathbf{v}_\mathbf{L}. \tag{2.3}$$

The coefficient $\tilde{D}$ [of the order of κ (Refs. 11 and 12)] is the diffusion coefficient of vortex lines.

The natural generalization of the production term σ_L in Eq. (2.1) would be to rewrite it in terms of rot $\mathbf{v}_s$ as

$$\sigma_L = -\beta\kappa L^2 + \left[\alpha_1 V + \frac{\beta_2}{\sqrt{2}}\sqrt{\kappa|\text{rot } \mathbf{v}_s|}\right] L^{3/2} - \left[\frac{\beta_1}{2}|\text{rot } \mathbf{v}_s| + \frac{\beta_4}{\sqrt{2}} V \sqrt{\frac{|\text{rot } \mathbf{v}_s|}{\kappa}}\right] L. \tag{2.4}$$

Equation (2.4) reduces to the right-hand side of Eq. (2.1) for pure rotation. Besides that, expressions (2.2) and (2.4) generalize Eq. (2.1) also on dynamical grounds. Note, indeed, that in Eq. (2.1) it is assumed that $|\text{rot } \mathbf{v}_s|$ is equal to 2Ω. However, it will take some time for $\mathbf{v}_s$ to get these values, by starting after some arbitrary initial state. Then, the form (2.1) will be useful after some transient interval, whereas Eqs. (2.2) and (2.4) are expected to be valid also for fast changes in $\mathbf{v}_s$. Further, Eq. (2.2) can be applied also in different situations, as plane Couette and Poiseuille flows. Thus, Eq. (2.2) with σ_L expressed by Eq. (2.4) is the central point of this paper, as it generalizes Eq. (2.1) both to a wider set of external conditions and to a wider domain of dynamical variations.

Comparison with a similar approach by Lipniacki[6] will be useful for a better understanding of both approaches. Lipniacki[6] has essentially proposed to use as variable the so-called "polarity vector" (see also Ref. 14), an important quantity in vortex dynamics, which he linked to the rotational of the averaged superfluid velocity

$$\mathbf{p} = \langle \mathbf{s}' \rangle = \frac{\int \mathbf{s}' d\xi}{\int d\xi} = \frac{\nabla \times \mathbf{v}_s}{\kappa L}, \tag{2.5}$$

averaged in a mesoscopic volume Λ. We do not enter into the specific details of this averaging, which may be found with more depth in Ref. 15 through the proposal of a Gaussian approximation for the distribution function of vortices. Note that in the transient interval when the turbulence has not a homogeneous distribution in the whole system, the polarity vector $\mathbf{p}$ depends on the spatial position of Λ in the system. On the other side, when the homogeneous situation is reached, any volume Λ in the system can be assumed to have the same polarity $\mathbf{p}$ (see Fig. 1).[15]

Note that $|\mathbf{p}| \in [0,1]$ measures the directional anisotropy of the tangent to the vortex lines: in particular, $|\mathbf{p}| = 1$ for a system of parallel vortices and $|\mathbf{p}| = 0$ for isotropic tangles. Thus, it is possible to express Eq. (2.4) in terms of $\mathbf{p}$ and to

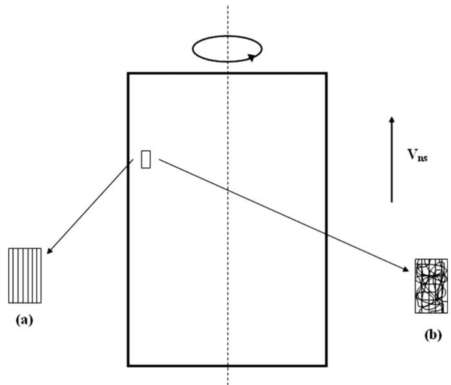

FIG. 1. When the stationary situation is reached, the polarization $\mathbf{p}$ assumes the same value in each small volume Λ in the container: (a) $|\mathbf{p}|=1$ in pure rotation and (b) $|\mathbf{p}|<1$ in the combined situation of rotation and counterflow.

group the terms in it in a slightly different way, namely, in two groups, one of them with the factor $VL^{3/2}$ and the other one with kL^2, mimicking in some way the form of the original Vinen's equation. In this way, we rewrite Eq. (2.4) as

$$\sigma_L = -\beta\kappa L^2\left[1-\frac{\beta_2}{\sqrt{2}\beta}\sqrt{\frac{|\mathrm{rot}\,\mathbf{v}_s|}{\kappa L}}+\frac{\beta_1}{2\beta}\frac{|\mathrm{rot}\,\mathbf{v}_s|}{\kappa L}\right] + \alpha_1 VL^{3/2}\left[1-\frac{\beta_4}{\sqrt{2}\alpha_1}\sqrt{\frac{|\mathrm{rot}\,\mathbf{v}_s|}{\kappa L}}\right], \quad (2.6)$$

so that, recalling $\beta_1=\sqrt{2}\beta_2-2\beta$, Eq. (2.2) assumes the more compact form

$$\frac{\partial L}{\partial t}+\nabla\cdot\mathbf{J}^L=\sigma^L = \alpha_1 VL^{3/2}[1-A\sqrt{|\mathbf{p}|}] - \beta\kappa L^2[1-\sqrt{|\mathbf{p}|}][1-B\sqrt{|\mathbf{p}|}], \quad (2.7)$$

where $B=\frac{\beta_1}{2\beta}$ and $A=\frac{\beta_4}{\sqrt{2}\alpha_1}$. In Ref. 4 coefficient B was found to be 0.89 while coefficient A is not properly a constant but undergoes a small step from 1 to 1.004 at the first counterflow critical velocity V_{c1}. In this work, we neglect this step assuming $A=1$. For totally unpolarized tangles, $\mathbf{p}=\mathbf{0}$ and Eq. (2.7) reduces to Vinen's equation. The polarization comes from pinned vortex lines, which begin and end on the walls of the container. In rotating containers, a part of the vortices go from one end to the other of the system, more or less parallel to the angular velocity vector. Near the walls, the polarization is a little bit higher than in the bulk, because the proportion of pinned vortices is higher, with respect to closed loops, and Eq. (2.7) predicts a reduction in the rate of formation and destruction of vortex lines, as compared with the bulk.

For a general hydrodynamic description, the evolution equations for $\mathbf{v}_n$ and $\mathbf{v}_s$ are needed. In particular, the evolution of $\mathbf{v}_s$ is necessary to describe the evolution of rot $\mathbf{v}_s$ in Eq. (2.4). A set of equations frequently used are the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) equations,[1,16] which in an inertial frame are written as

$$\rho_n\frac{\partial\mathbf{v}_n}{\partial t}+\rho_n(\mathbf{v}_n\cdot\nabla)\mathbf{v}_n = -\frac{\rho_n}{\rho}\nabla p_n-\rho_s S\nabla T+\mathbf{F}_{\rm ns}+\eta\nabla^2\mathbf{v}_n, \quad (2.8)$$

$$\rho_s\frac{\partial\mathbf{v}_s}{\partial t}+\rho_s(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s = -\frac{\rho_s}{\rho}\nabla p_s+\rho_s S\nabla T-\mathbf{F}_{\rm ns}+\rho_s\mathbf{T}. \quad (2.9)$$

Here, p_n and p_s are effective pressures, defined as $\nabla p_n=\nabla p+(\rho_s/2)\nabla V^2$, $\nabla p_s=\nabla p-(\rho_n/2)\nabla V^2$, p the total pressure, S the entropy, η the dynamic viscosity of the normal component, and $\rho_s\mathbf{T}$ the vortex tension force, which vanishes for rectilinear vortices and for isotropic vortex tangles, but which may be relevant in other situations. In the situations considered in this paper, we will assume $\mathbf{T}=\mathbf{0}$.

To describe the motion we need an expression for $\mathbf{F}_{\rm ns}$, the mutual force between normal and superfluid components. The usual expression by Hall, Vinen, Bekarevich, and Khalatnikov[1] can be written, in terms of the polarity $\mathbf{p}$, as

$$\mathbf{F}_{\rm ns}=\alpha\rho_s\kappa L\left[\hat{\mathbf{p}}\times[\mathbf{p}\times(\mathbf{V}-\mathbf{v_i})]+\frac{\alpha'}{\alpha}\hat{\mathbf{p}}\times(\mathbf{V}-\mathbf{v_i})\right], \quad (2.10)$$

with α and α' being friction coefficient depending on temperature, and $\mathbf{v_i}$ the "self-induced velocity," which in the HVBK equations is approximated by

$$\mathbf{v_i}=\tilde{\beta}\nabla\times\hat{\mathbf{p}}. \quad (2.11)$$

The expression for $\mathbf{F}_{\rm ns}$ must be consistent with the dynamics of L. In Sec. IV we will explore how Eq. (2.10) should be modified in order to be consistent with the evolution Eq. (2.7), and in Sec. V we will combine the equations in an analysis of plane Couette and Poiseuille flows in steady conditions.

III. ROTATING COUNTERFLOW

In this section, we investigate the proposed Eq. (2.7) for a rotating superfluid helium inside a cylindric container in the absence and in presence of counterflow when the homogeneous situation is reached. In this case the vortex flux $\mathbf{J}^L$ can be neglected, so that any variation in L is linked to the production term σ^L; further the polarity vector $\mathbf{p}$ can be approximately assumed having the same value over the whole system, as pointed out below. Therefore, the only equation needs to describe the homogeneous situation is the new evolution Eq. (2.7) for L (with $\mathbf{J}^L=0$).

Moreover, we compare some results of our proposal with those of Lipniacki[6] and with the experimental data of Swanson *et al.*[7] These authors considered a rotating container filled of helium II with an external counterflow $\mathbf{V}$ parallel to the angular velocity $\mathbf{\Omega}$ of the container. For high angular velocities, they observed two critical counterflow velocities V_{c1} and V_c such that for $0\leq V\leq V_c$, the line density L is approximately independent on V, undergoing only a small step (about 0.4%) at the first critical velocity V_{c1}, whereas for $V\geq V_c$ the line density L grows with V^2. Here we will neglect the small variation of L at the first critical velocity V_{c1}, because our proposal reduces to Eq. (2.1) in this situation—which was already carried out in Ref. 4—and it is not necessary for the comparison with Lipniacki's proposal because the latter is valid only for $V \geq V_c$.

A. Pure rotation

First of all, we consider the simplest situation of a cylindric container rotating around its axis. It is known that when the angular velocity Ω exceeds a critical value Ω_c and the stationary state is reached, vortex lines parallel to the rotation axis are present whose number density follows the law $L=2\Omega/\kappa$. The presence of these vortices may be explained observing that when the container begins to rotate, the viscous normal fluid rotates with it, whereas the superfluid remains initially at rest, due to its vanishing viscosity. In this situation, the difference between $\mathbf{v}_n$ and $\mathbf{v}_s$ is zero along the rotation axis, but it is maximum near the walls of the container, that is, the counterflow velocity increases for increasing distance from the axis. In this way the remnant vortices, which are formed during the cooling of helium and which are pinned to the walls, are influenced by the counterflow velocity. This implies the growth of these vortices in agreement with the dynamical description proposed by Schwarz. According to this idea, vortices will grow near the walls, due to the relative velocity between normal and superfluid velocities, and will migrate toward the bulk of the system, forming in the stationary situation a regular array of vortices parallel to the rotation axis.

The presence of vortices couples the normal fluid and the superfluid through the mutual friction force so that vortices are dragged by the normal fluid, and the average superfluid velocity $\mathbf{v}_s$ becomes different from zero. This fact justifies the relation $\nabla \times \mathbf{v}_s = 2\mathbf{\Omega}$ and the substitution of $2\Omega/\kappa L = |\mathbf{p}|$ in Eq. (2.1). At the light of the new arguments, in the case of pure rotation the vortex-line density becomes $L=2\Omega/\kappa$, which implies $|\mathbf{p}| \equiv 1$.

Consider now Eq. (2.7) in the case of pure rotation when the stationary solution is reached, that is $V=|\langle \mathbf{v}_n - \mathbf{v}_s \rangle| \approx 0$. In this case Eq. (2.7) has two stationary solutions, $|\mathbf{p}|=1$ and $|\mathbf{p}|=1/B^2$. As one can easily verify, the solution $|\mathbf{p}|=1$ is stable if $B<1$ and this is the case because the coefficient B was found to be 0.89.[4] To describe the nonstationary regime, one needs to use Eqs. (2.8) and (2.9) for the averaged normal and superfluid velocities.

B. Fast rotation and external counterflow

In this situation Eq. (2.7), together with the HVBK equations, should be valid also in transient situation, even if numerical simulation and experiments are needed.

In the hypothesis of coarse-grained adopted in this paper, we can assume that, after some transient time, the vortex tangle is homogeneous; this implies that the small volume Λ, used to define the polarity vector, has the property that it does not depend on the position vector $\mathbf{x}$ (see Fig. 1), so that the value of $\mathbf{p}$ is approximately the same everywhere.

The polarity vector $\mathbf{p}$ is parallel to the direction of rotation and external counterflow, and its modulus depends on the counterflow velocity. Looking at the definition of the vector $\mathbf{p}$, one notes that $|\mathbf{p}|=1$ for $V<V_c$, because $L \approx 2\Omega/\kappa \approx |\mathrm{rot}\, \mathbf{v}_s|$ in this situation, whereas $|\mathbf{p}|<1$ for $V>V_c$ because $|\mathrm{rot}\, \mathbf{v}_s| = 2\Omega$ and the vortex-line density is higher than $2\Omega/\kappa$ (see Fig. 2).

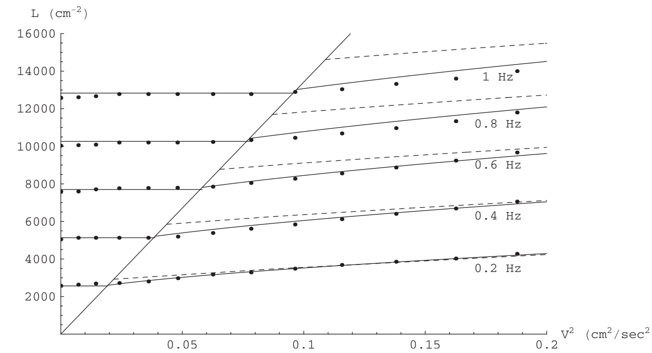

FIG. 2. Comparison of the stationary solutions of Lipniacki's model (3.10) (dashed line) and Jou and Mongiovì's model (2.7) (black line) with the experimental data (solid circles) by Swanson *et al.* for counterflow velocity bigger than the second critical velocity V_c and angular velocities 0.2, 0.4, 0.6, 0.8, and 1 Hz. Lipniacki's model does not give the horizontal part of the plot, corresponding to $V<V_c$.

When the homogeneous situation in the system is reached, Eq. (2.7) can be written as

$$\frac{dL}{dt} = L^{3/2}(1-\sqrt{|\mathbf{p}|})[\alpha_1 V - \beta\kappa L^{1/2}(1-B\sqrt{|\mathbf{p}|})]. \tag{3.1}$$

The nonzero stationary solutions of Eq. (3.1) are

$$|\mathbf{p}| = 1 \tag{3.2a}$$

and

$$L^{1/2} = \frac{\alpha_1}{\beta\kappa}V + B\sqrt{\frac{|\nabla \times \mathbf{v}_s|}{\kappa}}. \tag{3.2b}$$

To study the stability of the solution $|\mathbf{p}|=1$, we linearize Eq. (3.1) for the perturbations. In the hypothesis that the perturbation δ does not modify the vorticity $\vec{\omega} = \mathrm{rot}\, \mathbf{v}_s$, the relation $\delta|\mathbf{p}| = -(|\mathbf{p}|/L)\delta L$ is obtained, which allows us to obtain the following evolution equation for the perturbation δL

$$\left(\frac{\partial \delta L}{\partial t}\right)_{|\mathbf{p}|=1} = \left[\frac{\alpha_1 V}{2L^{1/2}} - \frac{1}{2}\beta\kappa(1-B)L\right]\delta L. \tag{3.3}$$

From the previous equation it follows that the solution $|\mathbf{p}|=1$ is stable for V less than

$$V_c = \frac{\beta}{\alpha_1}(1-B)\sqrt{|\nabla \times \mathbf{v}_s|\kappa}, \tag{3.4}$$

which corresponds to the critical velocity V_c in the experiments of Swanson *et al.*[7] Note that if $B=1$ in Eq. (3.4), the critical counterflow velocity for which the straight vortex lines parallel to the rotation axis become unstable is zero. From an experimental point of view this is not the case because a nonvanishing critical velocity is observed, confirming the value, $B=0.89<1$, obtained in Ref. 4.

For counterflow velocity higher than the critical velocity (3.4), the solution $|\mathbf{p}|=1$ becomes unstable, and the line density L assumes the value (3.2b) which depends on V and $|\text{rot } \mathbf{v}_s|$.

Now, we consider the second term in the right-hand side of Eq. (3.2b), namely, $B\sqrt{\frac{|\nabla\times\mathbf{v}_s|}{\kappa}}$. For low values of the counterflow velocity, the vorticity is essentially due to the rotation, and therefore we put $|\nabla\times\mathbf{v}_s|=2\Omega$, recovering the results obtained in Ref. 4.

C. Comparison with Lipniacki's proposal

Recently a hydrodynamical model of superfluid turbulence was proposed by Lipniacki[6] mainly with the aim to studying the hydrodynamics of partially polarized tangles arising in rotating counterflow or in plane Couette flow. Thus, it is interesting to compare with his work, whose aims are similar to ours.

Lipniacki writes Vinen's equation as

$$\frac{dL}{dt} = \alpha L^{3/2} c_1(|\mathbf{p}|)\mathbf{I}\cdot\mathbf{V} - \beta\alpha_2 c_2^2(|\mathbf{p}|)L^2, \qquad (3.5)$$

where β is a constant of the order of κ and α the friction coefficient appearing in the expression of the mutual friction force; $\mathbf{I}$ is the binormal vector,

$$\mathbf{I} = \frac{\langle \mathbf{s}'\times\mathbf{s}''\rangle}{\langle|\mathbf{s}''|\rangle}, \qquad (3.6)$$

defined by Schwarz[10] to describe the polarization of the binormal $\mathbf{s}'\times\mathbf{s}''$, of the vortex lines, with $\mathbf{s}'$ and $\mathbf{s}''$ being the first and second derivatives of the curve $\mathbf{s}(\xi)$ describing a vortex line with respect to the arc-length ξ, $\mathbf{s}'$ the unit tangent along the line, and $\mathbf{s}''$ the curvature vector.

The coefficients c_1 and c_2^2 measure the average curvature and curvature squared of the tangle, respectively. They are given, according to the microscopic model by Schwarz,[10] by

$$c_1 = \frac{1}{\Lambda L^{3/2}}\int |\mathbf{s}''|d\xi, \quad c_2^2 = \frac{1}{\Lambda L^2}\int |\mathbf{s}''|^2 d\xi, \qquad (3.7)$$

where Λ is the volume on which one makes the averaging indicated in Eq. (3.7). Lipniacki proposes that c_1 and c_2^2 depend on the polarization $|\mathbf{p}|$, and that they vanish for completely polarized tangles because in this case, $\mathbf{s}''=0$ for all the vortex lines. To describe the reduction in c_1 and c_2^2 with respect to its usual variable for a nonpolarized tangle, which will be designed as c_{10} and c_{20}^2, respectively, he assumes that

$$c_1(|\mathbf{p}|) \simeq c_{10}[1-|\mathbf{p}|^2], \quad c_2^2(|\mathbf{p}|) \simeq c_{20}^2[1-|\mathbf{p}|^2]^2. \quad (3.8)$$

In contrast, our expression (2.7) could be interpreted in this perspective as

$$c_1(|\mathbf{p}|) \simeq c_{10}[1-\sqrt{|\mathbf{p}|}], \quad c_2^2(|\mathbf{p}|) \simeq c_{20}^2[1-\sqrt{|\mathbf{p}|}][1-B\sqrt{|\mathbf{p}|}]. \qquad (3.9)$$

Therefore, it raises the question of the comparison of both Eqs. (2.7) and (3.5) with the experimental data, and a deeper understanding of the influence of polarity on the coefficients c_1 and c_2^2.

The evolution equation for the vortex-line density L, proposed by Lipniacki,[6] has the explicit form

$$\frac{dL}{dt} = \tilde{\alpha} I_0 c_{10} V L^{3/2}[1-|\mathbf{p}|^2] - \tilde{\beta}\alpha c_{20}^2 L^2[1-|\mathbf{p}|^2]^2, \qquad (3.10)$$

where $I_0=\mathbf{I}\cdot\hat{\mathbf{V}}$, and the subscript 0 stands for independence of I_0 on V and L. The author chooses for I_0 the same values found in pure counterflow, in such a way to not consider the anisotropy of the vortex tangle, which is present owing to the high values of rotation considered in the experiments by Swanson *et al.*[7]

Equation (3.10), as the author remarks, does not describe any of the two critical velocities, V_{c1} or V_c, of the experiments of Swanson *et al.*[7] Lipniacki's aim is instead to describe the relation between angular velocity, counterflow, and line-length density for polarized tangles above the second critical velocity V_c. This implies the need of a comparison, in the uniform steady rotation and counterflow, between Eqs. (2.7) and (3.10), and the experimental data of Swanson *et al.*[7]

The stationary solutions of the Eq. (3.10) are $|\mathbf{p}|=1$ (which however is unstable) and

$$L = \frac{L_H}{(1-(L_\omega/L)^2)^2}, \qquad (3.11)$$

where

$$L_H = V^2\left(\frac{c_{10}I_0}{\beta c_{20}^2}\right)^2 \quad \text{and} \quad L_\omega = \frac{|\text{rot } \mathbf{v}_s|}{\kappa} = \frac{2\Omega}{\kappa} \qquad (3.12)$$

are the steady-state vortex-line density in pure counterflow and in pure rotation, respectively.

In Fig. 2, we compare the results of Eqs. (2.7) and (3.10) with the experimental data of the Fig. 2 of Swanson's experiments. It follows that Eq. (2.7) (black line) describes better the experimental data (solid circle) than Eq. (3.10) (dashed line), not only for $V>V_c$, but it also yields the horizontal branch of the results for $V<V_c$, which are not described by Eq. (3.10). Comparison with experimental data shows that in the considered range of values of V and Ω, Eq. (2.7) fits better the experimental results.

A reason for the difference between proposals (2.7) and (3.10) could be related not to the evaluation of the integrals in Eq. (3.7) but to a different microscopic interpretation of some terms in the evolution equation for L. Schwarz's derivation[10] is based on the dynamics of vortex breaking and reconnection, and its production and destruction terms tend to zero for completely polarized systems, as rightly pointed out by Lipniacki. However, the origin of the rotational terms in Eq. (2.1) could be completely different. It is known that in rotating superfluid helium, the vortices grow near the walls due to the rotation and drift toward the center of the system, where they find a repulsion due to other vortices. These forces are different from zero even for completely polarized vortices, in contrast to the terms from Eq. (3.7). It could then be that the vanishing of the terms in Eq. (2.7) as $1-\sqrt{|\mathbf{p}|}$ had a different physical origin than the vanishing proposal by

Lipniacki from a different model. These open questions stress the need of the inclusion of rotational effects in a more general version of Schwarz's derivation of Vinen's equation.

IV. THERMODYNAMIC ANALYSIS OF POLARIZED SUPERFLUID TURBULENCE

In this section, we will perform two modifications of the expression of the mutual friction force, as used in the HVBK model, which are necessary to incorporate the anisotropy of the vortex tangle and to ensure the thermodynamic consistency of the evolution equation for L and for $\mathbf{v}_s$, according to the formalism of linear irreversible thermodynamics.[9,17] Since Eq. (2.7) differs from the usual Vinen's equation, it is logical to ask how these modifications will change the form of $\mathbf{F}_{\rm ns}$. For the sake of simplicity, we will neglect here the contribution of the self-induced velocity in Eq. (2.10).

First, we will take into account the anisotropy of the tangle introducing the tensor $\mathbf{\Pi}=\mathbf{\Pi}^s+\mathbf{\Pi}^a$, studied in Refs. 9 and 14:

$$\mathbf{\Pi}^s \equiv \frac{3}{2}\langle \mathbf{U}-\mathbf{s}'\mathbf{s}'\rangle, \quad \mathbf{\Pi}^a \equiv \frac{3}{2}\frac{\alpha'}{\alpha}\langle \mathbf{W}\cdot\mathbf{s}'\rangle. \tag{4.1}$$

In this equation $\mathbf{s}'$ is the unit vector tangent to the vortex lines, $\mathbf{s}'\mathbf{s}'$ is the diadic product, $\mathbf{U}$ is the unit matrix, $\mathbf{W}$ is the Ricci third-order tensor, and the angular brackets stand for the average over vortex lines in a given volume. The tensor $\mathbf{\Pi}^s$ describes the orientation of the tangents $\mathbf{s}'$ of the vortex lines, and the tensor $\mathbf{\Pi}^a$—associated to an axial vector—describes the polarization; in other words, $\mathbf{\Pi}^a$ is related to the first-order moment of the orientational distribution function of $\mathbf{s}'$ and $\mathbf{\Pi}^s$ is related to second-order moment. As shown in Ref. 14, using tensor $\mathbf{\Pi}$, the mutual friction force can be written as

$$\mathbf{F}_{\rm ns} = -\alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V}. \tag{4.2}$$

If we suppose isotropy in the tangle, it results $\mathbf{\Pi}^s=\mathbf{U}$, $\mathbf{\Pi}^a=\mathbf{0}$ and one finds the usual expression

$$\mathbf{F}_{\rm ns} = -\frac{2}{3}\alpha\rho_s\kappa L\mathbf{V}. \tag{4.3}$$

The tensor $\mathbf{\Pi}$ in Eq. (4.2) allows one to deal under a same formalism an array of parallel straight vortices as well as an isotropic tangle, and also the intermediate situations.

Now, we follow the general lines of Refs. 9 and 18 to propose a modification to Eq. (4.2) with the aim to determine an evolution equation for $\mathbf{v}_s$ consistent with Eq. (2.7). According to the formalism of nonequilibrium thermodynamics one may obtain evolution equations for $\mathbf{v}_s$ and L by writing $d\mathbf{v}_s/dt$ and dL/dt in terms of their conjugate thermodynamic forces $-\rho_s\mathbf{V}$ and ϵ_V. The evolution Eq. (2.9) for $\mathbf{v}_s$, neglecting inhomogeneous contributions of pressure, temperature, and velocity, in an inertial frame, is written as

$$\rho_s\frac{d\mathbf{v}_s}{dt} = -\mathbf{F}_{\rm ns} = \alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V}. \tag{4.4}$$

However, in the right-hand side of Eq. (4.2), additional contributions must be included to make Eq. (4.4) thermodynamically consistent with Eq. (2.7).

In a way similar to that presented in Ref. 9, we write $d\mathbf{v}_s/dt$ and dL/dt in matrix form in the system (4.5). In it, we write the equation for L in the form given in Eq. (2.7) and by means of Onsager-Casimir reciprocity we obtain an additional contribution to the evolution equation for $\mathbf{v}_s$. The result is

$$\begin{pmatrix} \dfrac{d\mathbf{v}_s}{dt} \\ \dfrac{dL}{dt} \end{pmatrix} = L\begin{pmatrix} -\dfrac{1}{\rho_s}\alpha\kappa\dfrac{2}{3}\mathbf{\Pi} & \pm\dfrac{\alpha_1}{\rho_s}L^{1/2}(1-\sqrt{|\mathbf{p}|})\hat{\mathbf{V}} \\ -\dfrac{\alpha_1}{\rho_s}L^{1/2}(1-\sqrt{|\mathbf{p}|})\hat{\mathbf{V}} & -\dfrac{1}{\epsilon_V}L(1-\sqrt{|\mathbf{p}|})(1-B\sqrt{|\mathbf{p}|}) \end{pmatrix}\begin{pmatrix} -\rho_s\mathbf{V} \\ \epsilon_V \end{pmatrix}. \tag{4.5}$$

The sign ambiguity present in that equation comes in a natural way from the Onsager-Casimir reciprocity relation. Indeed, in Feynman-Vinen view, L is a scalar quantity which does not change under time reversal, unlike the superfluid velocity $\mathbf{v}_s$ which changes sign. According to Onsager-Casimir, this leads to antisymmetry of crossed coefficients, thus leading to the + sign. In Schwarz view, L possesses vectorial properties and it would change on time reversal, just like the superfluid velocity. This leads to the symmetry of the kinetic coefficients in the matrix in Eq. (4.5), i.e., to the − sign in the upper right-hand term. Below, we will directly take the minus sign, for the sake of a more direct comparison with the work by Lipniacki.

Therefore the equation for $d\mathbf{v}_s/dt$ becomes

$$\rho_s\frac{d\mathbf{v}_s}{dt} = \alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V} - \epsilon_V\alpha_1 L^{1/2}(1-\sqrt{|\mathbf{p}|})\hat{\mathbf{V}}. \tag{4.6}$$

The new term not contained in the evolution Eq. (4.4) for $\mathbf{v}_s$ is the coupling term between $d\mathbf{v}_s/dt$ and ϵ_V in the matrix in Eq. (4.5). Note that this term depends on the direction but not on the modulus of V. This class of terms is called dry-friction terms.

Observing that in the steady state (L, $|\text{rot}\,\mathbf{v}_s|$, and $\mathbf{V}$ constant) the solutions of vortex-line density Eq. (2.7) can be written as

$$L^{1/2} = \sqrt{\frac{|\mathrm{rot}\ \mathbf{v}_s|}{\kappa}}, \quad \text{for } 0 < V < V_c, \tag{4.7}$$

$$L^{1/2} = \frac{\alpha_1}{\beta\kappa}(V - V_c) + \sqrt{\frac{|\mathrm{rot}\ \mathbf{v}_s|}{\kappa}}, \quad \text{for } V > V_c, \tag{4.8}$$

and substituting them in Eq. (4.6), we obtain the following expression for the coupling force

$$\mathbf{F}_{\mathrm{coupl}} = -\epsilon_V\alpha_1\left[L^{1/2} - \sqrt{\frac{|\mathrm{rot}\ \mathbf{v}_s|}{\kappa}}\right]\hat{\mathbf{V}} = 0, \quad \text{for } V < V_c, \tag{4.9}$$

$$\mathbf{F}_{\mathrm{coupl}} = -\epsilon_V\alpha_1\left[L^{1/2} - \sqrt{\frac{|\mathrm{rot}\ \mathbf{v}_s|}{\kappa}}\right]\hat{\mathbf{V}} = \epsilon_V\frac{\alpha_1}{\beta\kappa}(V - V_c)\hat{\mathbf{V}}, \quad \text{for } V > V_c. \tag{4.10}$$

As a consequence, for $V < V_c$ the coupling force is absent (as in pure rotation) while, for $V > V_c$, when the array of rectilinear vortex lines becomes a disordered tangle, the additional term (4.9) appears. Indeed, in almost-steady state (L and $|\mathrm{rot}\ \mathbf{v}_s|$ constant), for $V < V_c$, Eq. (4.4) would be valid, with L expressed by Eq. (4.7), whereas, for $V > V_c$ it would become

$$\frac{d\mathbf{v}_s}{dt} = \alpha\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V} + \epsilon_V\frac{\alpha_1}{\beta\kappa\rho_s}(V - V_c)\hat{\mathbf{V}}, \tag{4.11}$$

with L expressed by Eq. (4.8). Summarizing, in steady states for $V < V_c$ the dry-friction force is absent, while it appears for $V > V_c$, when the array of rectilinear vortex lines becomes a disordered tangle. Thus V_c indicates the threshold not only of the vortex-line dynamics but also of the friction acting on the velocity $\mathbf{v}_s$ itself; this seems logical, as both variables are mutually related.

Summarizing, in this section we have proposed to substitute the expression (4.3) of the mutual friction force used in the HVBK model with

$$\mathbf{F}_{\mathrm{ns}} = -\alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V} - \epsilon_V\alpha_1 L^{3/2}(1 - \sqrt{|\mathbf{p}|})\hat{\mathbf{V}} \tag{4.12}$$

for the sake of thermodynamic consistency with Eq. (2.7).

To complete the comparison between Lipniacki's and our model, we analyze in both models the expression of the mutual friction force, which in HVBK equation is expressed by Eq. (2.10), while in general terms it is expressed as

$$\mathbf{F}_{\mathrm{ns}} = \alpha\rho_s\kappa L\langle\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{V} - \mathbf{v}_i)]\rangle + \alpha'\rho_s\kappa L\langle\mathbf{s}'\times(\mathbf{V} - \mathbf{v}_i)\rangle. \tag{4.13}$$

Lipniacki neglects the coefficient α' in Eq. (4.13) and he approaches the quantity $\langle\mathbf{s}'\times(\mathbf{s}'\times\mathbf{V})\rangle \simeq [\langle\mathbf{s}'\mathbf{s}'\rangle - \mathbf{U}]\mathbf{V} = \mathbf{I}_v - \mathbf{V}$ (where $\mathbf{I}_v = \langle\mathbf{s}'(\mathbf{s}'\cdot\mathbf{V})\rangle$) with

$$\langle\mathbf{s}'\times(\mathbf{s}'\times\mathbf{V})\rangle \simeq \mathbf{p}\times(\mathbf{p}\times\mathbf{V}) - \frac{2}{3}(1 - |\mathbf{p}|^2)\mathbf{V}, \tag{4.14}$$

and the quantity $\langle\mathbf{s}'\times(\mathbf{s}'\times\mathbf{v}_i)\rangle \simeq \tilde{\beta}\langle\mathbf{s}'\times\mathbf{s}''\rangle = \tilde{\beta}c_1 L^{1/2}\mathbf{I}$ with

$$\langle\mathbf{s}'\times(\mathbf{s}'\times\mathbf{v}_i)\rangle \simeq -\tilde{\beta}I_0 c_{10}(1 - |\mathbf{p}|^2)L^{1/2}\hat{\mathbf{V}}. \tag{4.15}$$

In explicit terms he uses

$$\mathbf{F}_{\mathrm{ns}} = \alpha\kappa\rho_s L\left[\mathbf{p}(\mathbf{p}\cdot\mathbf{V}) - \mathbf{V}\frac{2 + |\mathbf{p}|^2}{3} + \beta I_0 c_{10}(1 - |\mathbf{p}|^2)L^{1/2}\hat{\mathbf{V}}\right]. \tag{4.16}$$

So in the work of Lipniacki, the tensor $\frac{2}{3}\mathbf{\Pi}^s = \langle\mathbf{U} - \mathbf{s}'\mathbf{s}'\rangle$ assumes the expression

$$\frac{2}{3}\mathbf{\Pi}^s \simeq [\mathbf{U} - \mathbf{pp}] + \frac{2}{3}(1 - |\mathbf{p}|^2)\mathbf{U} = \frac{5 - 2|\mathbf{p}|^2}{3}\mathbf{U} - \mathbf{pp}. \tag{4.17}$$

Note that Eq. (4.17) does not respect the relation trace$[\langle\mathbf{U} - \mathbf{s}'\mathbf{s}'\rangle] = 2$, following from the normalized character of $\mathbf{s}'$, if $|\mathbf{p}| \neq 1$. In fact it is

$$\mathrm{trace}\left[\frac{5 - 2|\mathbf{p}|^2}{3}\mathbf{U} - \mathbf{pp}\right] = 5 - 3|\mathbf{p}|^2. \tag{4.18}$$

The last term in Eq. (4.16) is a consequence of the drift of the tangle in the direction of the counterflow, caused by its anisotropy, where $\mathbf{I} = I_0\hat{\mathbf{V}}$. This term is substituted in our model by the last term in Eq. (4.6), which we can rewrite, recalling that $\epsilon_V = \rho_s\kappa\tilde{\beta}$ and $\alpha_1 = \alpha c_{10}I_0$ as

$$\mathbf{F}_{\mathrm{coupl}} = -\epsilon_V\alpha_1 L^{3/2}(1 - \sqrt{|\mathbf{p}|})\hat{\mathbf{V}} = -\rho_s\kappa\tilde{\beta}\alpha c_{10}I_0 L^{3/2}(1 - \sqrt{|\mathbf{p}|})\hat{\mathbf{V}}. \tag{4.19}$$

As it is seen, this term differs from the one of Lipniacki, in the contribution due to the polarization of the tangle, which in our approach depends on $1 - \sqrt{|\mathbf{p}|}$, and in Lipniacki's one on $1 - |\mathbf{p}|^2$. We note also that, in this interpretation, we must choose the negative sign in the expression of this coupling term, in agreement with the microscopic derivation of the filament model by Schwarz.

Lipniacki does not consider the tension $\mathbf{T}$. For a comparison with our work, we must observe that in Lipniacki's model, the quantity $\langle\mathbf{s}'\mathbf{s}'\rangle$ is approximated by $\mathbf{pp}$, and this approximation is correct only if most of the vortex lines in the volume have the same direction.

In Ref. 14 we have provided a microscopic paramagnetic analogy to relate $\mathbf{p} = \langle\mathbf{s}'\rangle$ with $\mathbf{\Omega}$ and $\mathbf{V}$, in the case of simultaneous counterflow and rotation, but we have not studied the statistic of the curvature vector $\mathbf{s}''$. In contrast, Lipniacki leaves open the value of $\mathbf{p}$ and makes some simple hypotheses about $\langle|\mathbf{s}''|\rangle$ and $\langle|\mathbf{s}''|^2\rangle$ in his analysis of the possible influence of polarization in the Vinen's equation.

A further difference between our model and that of Lipniacki refers to the form of the vortex flux for which he writes

$$\mathbf{J}^L = L\mathbf{v}^L = L[\mathbf{v}_s + \alpha \mathbf{p} \times \mathbf{V} + \beta\alpha I_0 c_{10}(1 - |\mathbf{p}|^2)L^{1/2}\hat{\mathbf{V}} + \beta\alpha \mathbf{I}_k L^{1/2}], \quad (4.20)$$

where the vector $\mathbf{I_k}$ represents the curvature of $\vec{\omega}_s$ lines. This last term is exactly zero if the vortex lines are closed, isotropic of straight, and otherwise it is expected to be small, except for the case when all the vortex lines are parallel to each other but bent. This is only the convective contribution, to which it should be added the diffusive contribution $\mathbf{J}^L = -\tilde{D}\nabla L$.

V. VORTEX-LINE DENSITY IN STEADY PLANE FLOWS

In the relation (2.4) [and Eq. (2.7)] we have rewritten the right-hand side of Eq. (2.1) for rotating counterflow turbulence in liquid helium in terms of $|\text{rot } \mathbf{v}_s|$. For pure rotation, $|\text{rot } \mathbf{v}_s| = 2\Omega$ and we just have our original equation, but Eq. (2.4) may be also used to describe situations with barycentric motion as plane Couette and Poiseuille flows (without external heat flux) between two parallel plates. Here we will consider two plates separated by a distance D, one at rest and the other one moving at velocity $\mathbf{V}_0$ (Couette flow), or plane Poiseuille flow, given by a longitudinal pressure gradient along the direction of two parallel quiescent walls. Here, we will deal with steady states and quasistationary states. We will assume that the flow of the normal component remains laminar. This requires that the Reynolds number DV_0/η, with η as the viscosity of the normal component and V_0 the characteristic velocity of flow, is sufficiently small. On the other side, in analogy with the rotating container, we assume that the velocity V_0 is sufficiently high to neglect the "effects of the walls."[19] The essential problem in both cases is to find the distribution of the superfluid velocity, the vortex density and the mutual friction force. We will focus our attention mainly to steady-state situations, as simple illustration of the changes implied by the new Eqs. (2.7) and (4.6), for L and $\mathbf{v}_s$.

A. Plane Couette flow

We assume two plane surfaces at $z=0$ and $z=D$ such that the second one moves parallel to the first one at the velocity $\mathbf{V}_0$, and that the relative velocity between normal and superfluid velocities has a profile $\mathbf{V} = [V_x(z), 0, 0]$. The dynamics of vortex formation is similar to that in the rotating cylinder. When the upper plate starts suddenly moving with respect to the lower plate, the normal component starts moving under the action of the viscous force and nonslip condition, whereas the superfluid component will remain initially insensitive to the motion of the plate. Thus, a relative velocity (the counterflow velocity) $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$ will arise between the two components. This counterflow velocity $\mathbf{V}$ depends on the distance from the lower plate, in particular $\mathbf{V}$ is maximum for $z=D$ (upper plate) and zero for $z=0$ (lower plate).

When the counterflow velocity reaches a critical value near the moving plane, the remnant vortices, always present in He II, begin to lengthen and reconnect to form other vortices, which diffuse toward the lower plate (at rest) forming, in the stationary situation, an array of vortices parallel to each other and to the plates and orthogonal to the flow. Through the vortices, the normal and the superfluid components become coupled by the mutual friction force $\mathbf{F}_{\text{ns}}$, and the superfluid will tend to match its velocity with that of the normal fluid ($V \to 0$); this will introduce a rot $\mathbf{v}_s \neq 0$ in the superfluid, expressed by $|\partial \mathbf{v}_s / \partial z|$. After a sufficiently long time, it is expected that a steady shear flow will have formed, with $\mathbf{v}_n = \mathbf{v}_s$ depending only on z and having the x direction and such that $\partial \mathbf{v}_n / \partial z = \partial \mathbf{v}_s / \partial z = \mathbf{V}_0 / D$, corresponding to the physical Newtonian linear profile, which follows from Eqs. (2.8) and (2.9) with vanishing tension force $T=0$, and Eq. (4.12) in which $\mathbf{F}_{\text{ns}} = \mathbf{0}$ for $\mathbf{V} = \mathbf{0}$ and $|\mathbf{p}| = 1$. Then, it results $|\text{rot } \mathbf{v}_s| = \mathbf{V}_0 / D$.

Introduction of this value in Eq. (2.4) would give the areal density of parallel and straight vortex lines, perpendicular to the flow. However, as it has been said in Sec. II, the replacement of $\mathbf{\Omega}$ in terms of rot $\mathbf{v}_s$ is deeper than a formal substitution because $\mathbf{v}_s$ will not become related to the gradient of the barycentric velocity until a complex transient process has lapsed. Thus, the direct replacement of 2Ω in Eq. (2.1) by dv_{sx}/dz in shear flows, with v_{sx} the x component of the macroscopic superfluid velocity, will be valid for steady states and for relatively slow variations with respect to steady states. Otherwise, rot $\mathbf{v}_s$ with its own nontrivial dynamics should be considered in Eq. (2.4). The situation of Couette flow may be rather illustrative of these features.

Then, the dynamics of L in this case is described by

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha_1 V + \beta_2 \sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}\right] L^{3/2} - \left[\frac{\beta_1}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right| + \beta_4 V \sqrt{\frac{1}{2\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}\right] L - \nabla \cdot \mathbf{J}^L, \quad (5.1)$$

where the coefficients should obey the relations indicated below Eq. (2.1), and where the last term stands for the effects of the vortex flux for inhomogeneous systems.

In the stationary situation $V \approx 0$ and, according to Eq. (5.1), there will be a completely polarized array of vortices, perpendicular to the velocity and to the velocity gradient, with uniform areal density given by

$$L = \frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right| = \frac{V_0}{\kappa D}. \quad (5.2)$$

In this view, the stationary character of L would require V to be zero, in such a way that normal fluid, superfluid, and vortices would move at the same speed and without internal friction. However, Eq. (5.1) has the intrinsic feature that for V less than a value V_c, the vortex-line density does not depend on V and is given by Eq. (5.2). This critical relative velocity is, according to Eq. (5.1),

$$V_c = \frac{\beta}{\alpha_1}\left[2\frac{\beta_4}{\alpha_1} - \frac{\beta_2}{\beta}\right]\sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|} \cong c' \sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}, \quad (5.3)$$

with $c' \approx 3.7$, according to the values of the coefficients used in Eq. (2.1) to describe the value of V_c in rotating counterflow velocity.

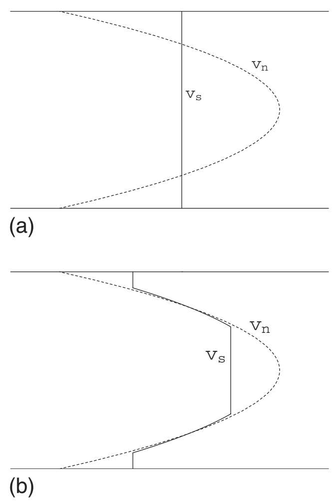

FIG. 3. (a) Initial profile and (b) steady profile of the superfluid (continuous line) and normal velocities (dashed line).

This indicates that the ordered array of vortices formed in the Couette flow is stable until $V < V_c$. This means that, as V_0 grows, the regular array of rectilinear vortices, orthogonal to V_0, is still present and the velocities $\mathbf{v}_n$, $\mathbf{v}_s$, and $\mathbf{V}$ have rectilinear profiles, but with slightly different slope. The value of V is maximum near the plane $z=D$. When the counterflow velocity V reaches the critical value V_c, the regular Couette array of vortices becomes unstable and a disordered tangle of vortex lines appears between the two plates in the zone. If one did not apply Eq. (2.1), but only intuitive reasoning without the detailed quantitative analysis showing this critical velocity, one would expect that for $V > 0$, there will always be a disordered tangle of vortices.

B. Plane Poiseuille flow

Equation (5.1) may be applied to plane Poiseuille flow between two quiescent parallel walls at $z = \pm D/2$, driven by a longitudinal pressure gradient. The steady velocity profile for a Newtonian viscous fluid is parabolic, and has the form

$$V_x(z) = \frac{\Delta p}{2\eta l}\left[\frac{D^2}{4} - z^2\right] = \frac{\Delta p}{\eta l}\frac{D^2}{8}\left[1 - \frac{4z^2}{D^2}\right] = V_{\max}\left[1 - \frac{4z^2}{D^2}\right], \tag{5.4}$$

with $\frac{\Delta p}{l}$ the pressure gradient, η the viscosity, and $V_{\max} = (D^2\Delta p)/(8\eta l)$. The pressure gradient acts on each component in the proportion established by the HVBK Eqs. (2.8) and (2.9).

Initially, the velocity profile of the normal component, submitted to viscous effects and to no-slip conditions on the walls, will be rather different from that of the superfluid component, which may slip freely along the walls [see Fig. 3(a)]. As a useful simplification, one may approximate the velocity profiles as a parabolic (Poiseuille profile) and a flat profile, respectively,[20] which equals one to each other in two points at the distance z_0 from the center of the plates. Then, one must search how these profiles will evolve under their mutual interaction due to the friction force, caused by the presence of the vortices.

During the transient regime, vortices will be produced mainly in the regions where the relative velocity V is higher than a critical value V_c—which may be also the central region—but they will be transferred to the matching region where $\mathbf{v}_n = \mathbf{v}_s$ because of the second term in expression (2.10) for the mutual friction force, which is a Magnus force yielding a vortex lateral drift velocity described by $\mathbf{v}_{L(\text{lateraldrift})} = \alpha' \mathbf{s}' \times \mathbf{V}$. The accumulation of vortices in the region where the two fluids have the same velocity will enlarge the width of the matching region (the profile of $\mathbf{v}_s$ tends to the profile of $\mathbf{v}_n$), until arriving at a situation where V will be lower than V_c so that not more vortices will be produced. The steady profile will have the approximate form of Fig. 3(b), similar to that considered by Samuels (Fig. 7 of Ref. 21), but in the matching region $\mathbf{v}_n$ and $\mathbf{v}_s$ are not exactly equal, in contrast with Couette flow or rotating cylinder, because there is need of a friction force to cancel out the term in the pressure gradient in the HVBK equations, as shown in Eq. (5.6) below.

In the steady state, for isothermal flow, and neglecting the tension $\mathbf{T}$, which vanishes for rectilinear vortices and for isotropic tangles, Eqs. (2.8) and (2.9) reduce to

$$-\frac{\rho_n}{\rho}\nabla p_n + \mathbf{F}_{\text{ns}} + \eta\nabla^2\mathbf{v}_n = 0, \tag{5.5}$$

$$-\frac{\rho_s}{\rho}\nabla p_s - \mathbf{F}_{\text{ns}} = 0. \tag{5.6}$$

By adding these equations one obtains $-\nabla p + \eta\nabla^2\mathbf{v}_n = 0$, which shows that the velocity profile of the normal component is the usual one corresponding to the motion it would have by itself, without the interaction with the superfluid unless some contributions with $\mathbf{T} \neq 0$ would appear, in the form, for instance, of local anisotropy vortex tangles. On the other side, from Eq. (5.6) it is seen that $\mathbf{F}_{\text{ns}}$ will be different from zero, given by $\mathbf{F}_{\text{ns}} = -\frac{\rho_s}{\rho}\nabla p_s$. Thus, $\mathbf{v}_n$ and $\mathbf{v}_s$ will be slightly different, if ∇p is low enough, and there will be an array of straight vortices, which we calculate below.

The most relevant features of the steady profile are: the width $2z_c$ of the central zone without vortices and flat $\mathbf{v}_s$ profile, the width z_w of the boundary layer also without vortices and flat $\mathbf{v}_s$ profile, and $\bar{L}$, the averaged vortex density in the matching zone where vortices concentrate. We will compute them from simple qualitative arguments.

To compute z_c and z_w we will ask that the corresponding circulation of V_{ns} in these regions is lower than the vorticity quantum κ. This is a sufficient condition for the lack of vortices in this zone. The argument is similar to that which could be used to estimate the critical angular velocity for the formation of the first vortex line in a rotating cylinder. If the cylinder is rotating with angular speed Ω, the circulation of V will be $2\pi R^2\Omega$; to obtain Ω_c one equates this quantity to κ, and one gets $\Omega_c = \kappa/(2\pi R^2)$. The exact result is $\Omega_c = \kappa \ln(b/a_0)/(2\pi R^2)$ (Ref. 1), with a_0 the radius of the vortex line and b a distance of the order of the line spacing, which in the case of the one vortex is of the order of the radius R of the cylinder. In the plane Poiseuille flow situation the quantity b is of the order z_c, in the central zone, and of the order z_w, in the boundary layer zone.

Thus, to estimate z_c we calculate the circulation of $V_{ns} = \frac{\Delta p}{2\eta l}[z_c^2 - z^2]$ in the zone between $z=0$ and $z=z_c$ and equate it to $\kappa \ln(b/a_0)$. One has

$$\Gamma_c = \oint_\gamma V_{ns} \cdot dl = - \int_0^{z_c} \left(\frac{\Delta p}{2\eta l}[z_c^2 - z^2] \right) \bigg|_{z=0} dx$$

$$= \frac{\Delta p}{2\eta l} z_c^3 \approx \kappa \ln(c z_c / a_0), \tag{5.7}$$

where γ is the contour of the square whose side is z_c and c is a numerical constant of the order of the unity. This may be expressed in terms of the maximum velocity V_{max} of $\mathbf{v}_n$ as given by Eq. (5.4), leading to expression

$$\frac{z_c^3}{D^3} = \frac{\kappa \ln(c z_c / a_0)}{4 D V_{max}}. \tag{5.8}$$

Concerning the width of the boundary layer z_w, a similar argument yields

$$\Gamma_w = \oint_{\gamma_1} V_{ns} \cdot dl = \int_0^{z_w} \left(\frac{\Delta p}{2\eta l}\left[\left(\frac{D}{2} - z_w \right)^2 - z^2 \right] \right) \bigg|_{z=D/2} dx$$

$$= \frac{\Delta p}{2\eta l}[D z_w^2 - z_w^3] \approx \kappa \ln(c' z_w / a_0), \tag{5.9}$$

where γ_1 is the contour of the square whose side is z_w and c' is a numerical constant of the order of the unity. Up to second order in z_w, this yields

$$\frac{\Delta p}{2\eta l} D z_w^2 = \kappa \ln(c' z_w / a_0), \tag{5.10}$$

and using expression (5.4) for the $\mathbf{v}_n$ profile, the previous expression may be rewritten in terms of V_{max} as

$$\frac{z_w^2}{D^2} = \frac{\kappa \ln(c' z_w / a_0)}{4 D V_{max}}. \tag{5.11}$$

This expression is similar to the one obtained by Samuels[21] for the width of the outer layer in a cylindrical Poiseuille flow in a tube of diameter D [his Eq. (15)], which was

$$\left(\frac{r_c}{D} \right)^2 = \frac{\kappa}{8\pi D V_{max}} \ln\left(\frac{8 r_c}{a_0} \right). \tag{5.12}$$

From Eqs. (5.8) and (5.11) it is found that the widths z_c and z_w decrease for increasing V_{max} as $z_c \sim V_{max}^{-1/3}$ and $z_w \sim V_{max}^{-1/2}$. Thus for increasing V_{max} (i.e., increasing pressure gradient) the central zone and the outer zone-boundary layers free of vortices will become thinner. The flat profile of $\mathbf{v}_s$ in these zones is consistent with the absence of vortices, according to the relation $L = |\partial \mathbf{v}_s / \partial z| / \kappa$, analogous to the expression (5.2), and which vanishes for flat profile.

In the matching region the value of $\mathbf{v}_n - \mathbf{v}_s$ will not be strictly zero, but because of restriction (5.6) if $\mathbf{v}_n - \mathbf{v}_s$ is approximately constant in this region, one will have that

$$L(z) = \frac{1}{\kappa} \left| \frac{\partial \mathbf{v}_s}{\partial z} \right| \approx \frac{1}{\kappa} \left| \frac{\partial \mathbf{v}_n}{\partial z} \right| = \frac{8 V_{max}}{\kappa D^2} |z|. \tag{5.13}$$

It is known that there exist two values of z where the velocities, $\mathbf{v}_n$ and $\mathbf{v}_s$, are equal, but, in general, in the rest of the z domain they could not be exactly equal. This implies that the mutual friction force could depend on z and that the distribution of the vortices could be inhomogeneous. To overcome this problem, we average the value of L in the region between $z=z_c$ and $z=z_1=D/2-z_w$. Of course, the value of $\bar{L}$ in the region between $z=-z_1=-D/2+z_w$ and $z=-z_c$ will be the same of the first region by symmetry. To estimate, we assume that the averaged profile of the superfluid velocity can be approximated by the profile of the normal velocity, so obtaining

$$\bar{L} = \frac{8 V_{max}}{\kappa D^2} \left| \frac{z_c - z_1}{2} \right| = \frac{2 V_{max}}{\kappa D} \left| 1 + 2\left(\frac{z_c}{D} - \frac{z_w}{D} \right) \right|. \tag{5.14}$$

Introducing z_c and z_w as obtained from Eqs. (5.8) and (5.11) we would have an estimate of $\bar{L}$ in terms of V_{max}, or, equivalently, in terms of Δp. A more detailed analysis could be carried out from the transversal terms of the vortex flux, where the Magnus drift and the diffusion flux in Eq. (2.3) would cancel each other.

Expression (5.8) may be used to obtain the conditions for a laminar flow without any vortex. This situation will be found when the width of the central zone without vortices z_c is bigger than $D/2$. This leads to the condition $D V_{max}/\kappa \leq 2 \ln(D/(2a_0))$. For $D \approx 10^{-2}$ m, and since $a_0 \approx 10^{-10}$ m, we have $D V_{max}/\kappa \leq 20$. The dimensionless quantity $D V_{max}/\kappa$ is analogous to the Reynolds number. In viscous fluid, the Reynolds number is defined as DV/ν, with ν being the kinematic viscosity $\nu = \eta/\rho$, which has dimensions $m^2\, s^{-1}$. The vorticity quantum κ has also dimension $m^2\, s^{-1}$ and therefore $D V_{max}/\kappa$ may be considered as a quantum Reynolds number. A similar number has been used in pure counterflow experiments in cylindrical containers of diameter D where, for instance, the appearance of the first vortex takes place at $T = 1.7$ K for $DV/\kappa \approx 80$.[22] A more rigorous stability analysis would be convenient to obtain more values of the critical quantum Reynolds number in both situations.

VI. CONCLUSIONS

The quantized character of vorticity in superfluids emphasizes the special importance of vortex lines, whose dynamics becomes a central aspect of rotating or turbulent flows of superfluids. The main proposal of this paper is Eq. (2.4) for the evolution of vortex-line density, which generalizes our previous proposal (2.1) for rotating counterflow situations. Here, by writing the local average rotational of the superfluid component instead of the angular velocity, we have enlarged the set of applications of the theory in two main aspects. One of them is that Eq. (2.4), in contrast to Eq. (2.1), may be applied not only to rotation but also to shear flows, as illustrated in Sec. V. The second enlargement is of dynamical nature: in Eq. (2.4) rot $\mathbf{v}_s$ appears, and $\mathbf{v}_s$ itself should satisfy its own evolution equation, which is coupled to the evolution of $\mathbf{v}_n$, the velocity of the normal component. Then, Eq. (2.4) becomes deeply coupled to the HVBK Eqs. (2.8) and (2.9) for $\mathbf{v}_n$ and $\mathbf{v}_s$ not only through the mutual force $\mathbf{F}_{ns}$ between the normal and the superfluid components, which requires the knowledge of L, but also because in Eq. (2.4), $\mathbf{v}_s$ is needed to obtain L. Thus, the coupling of these equations is much emphasized in Eq. (2.4) as compared to Eq. (2.1).

For situations close to nonequilibrium steady states or for slow variations of $\mathbf{v}_s$, in such a way that rot $\mathbf{v}_s$ is well described by the angular velocity or by the barycentric velocity gradient, Eqs. (2.1) or (5.1) describe the vortex-line density in terms of Ω or dv_{sx}/dz. In each case we have provided an estimation of the vortex density and of the superfluid velocity profile in the steady state.

We have compared our proposal with that of Lipniacki, which shares the objectives of the present paper but stresses the polarization $\mathbf{p}=\text{rot } \mathbf{v}_s/kL$ more than rot $\mathbf{v}_s$ itself. Lipniacki's evolution equation for L is, essentially, the classical Vinen's equation, but with the new aspect that its coefficients become dependent on the polarization $\mathbf{p}$. The disagreement between our Eq. (2.7) and the Lipniacki's proposal (3.10) may be due to the different physical origin of the terms dependent on the polarization. Our opinion is that Schwarz derivation of Vinen's Eq. (3.5) does not include some relevant contributions of rotational systems. For straight parallel vortices, as those arising in pure rotation experiments, both the production and the destruction terms vanish. This is consistent with Schwarz's postulates for the vortices, but in purely rotational flows the dynamics of vortices has a different origin, related to the migration of vortices formed on the wall toward the center of the system, and with repulsion forces among vortices. Thus a general treatment would require to include these effects besides the Scharwz effects in Eq. (3.5), and it could provide a further understanding of the differences between Eqs. (2.7) and (3.10). In any case, comparison with experimental results in Fig. 2 indicates that Eq. (2.7) yields a better description of them.

Another interesting example which draws the attention is the situation proposed in the paper by Eltsov *et al.*[23] There the authors report experimental, numerical, and theoretical studies on the propagation of a vortex front in a vortex-free region filled by rotating superfluid ^{3}He-B. After creating a vortex-free Landau state, where the superfluid component is at rest and the normal component moves with the rigid rotation Ω, they inject a seed vortex into the sample observing a rapid local evolution of the vorticity toward the equilibrium state (a regular array of vortices parallel to the rotation axis). This phenomenon could be explained in a qualitative way as follows: in the free-vortex region, the relative velocity $\mathbf{V}$ between normal and superfluid components is nonzero and it is orthogonal to the rotation axis whereas the mutual friction is zero because of the absence of vortices. The presence of vortices coupled the two components so a mutual friction force $\mathbf{F}_{ns}$ arises. Analyzing the general expression (4.13) of the mutual friction force onto the boundary between the vortex-free and the vortex states, we observe that the first term refers to the torsion of the tangle whereas the second is parallel to the direction of vortices wave (see Fig. 1 of his work). From Eq. (2.9), or Eq. (4.11), we deduce that the presence of the mutual friction force moves the superfluid component in such a way that the two components couple and the vorticity rot $\mathbf{v}_s$ becomes non-null.

This change in the state of ^{3}He-B from free-vortex state to vortex state is also described by the production term of Eq. (2.7) or (2.1), when a seed of vortex is inserted in the sample: in the vortex-free Landau state, the relative velocity is not zero, so the production term has two stable solutions, $L=0$ and $p=1$ (or $L=2\Omega/\kappa$),[4] which are both admissible. The injection of a vortex in the cylinder allows the system to change from $L=0$ to $L=2\Omega/\kappa$, after some transient regime. Of course, dynamical equations for $\mathbf{v}_n$, $\mathbf{v}_s$, and L [Eq. (2.7)] are needed for a complete dynamical description of the vortex tangle propagation.

After the propagation of the crest of vortices inside the cylinder, a homogeneous situation is reached. There, the polarity vector is uniformly distributed over the sample but it is not exactly 1. This behavior is caused by the fact that in this situation the relative velocity between superfluid and normal component is not exactly zero, that is, $\mathbf{v}_n-\mathbf{v}_s\neq 0$. When, instead, the steady state is reached the relative velocity $\mathbf{V}$ is zero so that the stationary solution will be $\mathbf{p}=1$, i.e., $L=2\Omega/\kappa$, as in a classical rotating helium II. The propagation of vortex fronts and the prediction of their speed of propagation could be an interesting physical situation to check in the future the merits of generalized equations for vortex dynamics.

ACKNOWLEDGMENTS

We acknowledge the support of the Acción Integrada España-Italia (Grant No. S2800082F HI2004–0316 of the Spanish Ministry of Science and Technology and Grant No. IT2253 of the Italian MIUR). D.J. acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Education under Grant No. FIS 2006–12296-C02–01 and of the Direcció General de Recerca of the Generalitat of Catalonia, under Grant No. 2005 SGR-00087. M.S.M. and M.S. acknowledge the financial support from "Fondi 60%" of the University of Palermo. M.S. acknowledges the "Assegno di ricerca: Studio della turbolenza superfluida e della sua evoluzione" of the University of Palermo.

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[1] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991).
[2] *Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (Springer, Berlin, 2001).
[3] S. K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. **67**, 37 (1995).
[4] D. Jou and M. S. Mongiovì, Phys. Rev. B **69**, 094513 (2004).
[5] W. F. Vinen, Proc. R. Soc. London, Ser. A **242**, 493 (1957).
[6] T. Lipniacki, Eur. J. Mech. B/Fluids **25**, 435 (2006).
[7] C. E. Swanson, C. F. Barenghi, and R. J. Donnelly, Phys. Rev. Lett. **50**, 190 (1983).
[8] N. Anderson, T. Sidery, and G. L. Comer, Mon. Not. R. Astron. Soc. **381**, 747 (2007).
[9] D. Jou and M. S. Mongiovì, Phys. Rev. B **72**, 144517 (2005).
[10] K. W. Schwarz, Phys. Rev. B **38**, 2398 (1988).
[11] M. S. Mongiovì and D. Jou, Phys. Rev. B **75**, 024507 (2007).
[12] D. Jou, M. S. Mongiovì, and M. Sciacca, Phys. Lett. A **368**, 7 (2007).
[13] M. Sciacca, M. S. Mongiovì, and D. Jou, Math. Comput. Modell. **48**, 206 (2008).
[14] D. Jou and M. S. Mongiovì, Phys. Rev. B **74**, 054509 (2006).
[15] S. K. Nemirovskii, Phys. Rev. B **57**, 5972 (1998).
[16] I. L. Bekarevich and I. M. Khalatnikov, Sov. Phys. JETP **13**, 643 (1961).
[17] S. K. Nemirovskii, G. Stamm, and W. Fiszdon, Phys. Rev. B **48**, 7338 (1993).
[18] S. K. Nemirovskii and V. V. Lebedev, Zh. Eksp. Teor. Fiz. **84**, 1729 (1983); [Sov. Phys. JETP **57**, 1009 (1983)].
[19] M. S. Mongiovì and D. Jou, Phys. Rev. B **72**, 104515 (2005).
[20] S. P. Godfrey, D. C. Samuels, and C. F. Barenghi, Phys. Fluids **13**, 983 (2001).
[21] D. C. Samuels, Phys. Rev. B **46**, 11714 (1992).
[22] M. S. Mongiovì and D. Jou, J. Phys.: Condens. Matter **17**, 4423 (2005).
[23] V. B. Eltsov, A. I. Golov, R. de Graaf, R. Hänninen, M. Krusius, V. S. L'vov, and R. E. Soltntsev, Phys. Rev. Lett. **99**, 265301 (2007).